On similarity and pseudo-similarity solutions of Falkner-Skan boundary layers

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Abstract. This paper deals with the two-dimensional incompressible, laminar, steady-state boundary layer equations.First, we determine a family of velocity distributions outside the boundary layer such that these problems may have similarity solutions.Then, We examine in detail new exact solutions, called *Pseudo-similarity*, where the external velocity varies inversely-linear with the distance in the x-direction along the surface ($U_e(x) = U_\infty x^{-1}$). The analysis shows that solutions exist only for a lateral suction. Here it is assumed that the flow is induced by a continuous permeable surface with the stretching velocity $U_w x^{-1}$. For specified conditions, we establish the existence of an infinite number of solutions, including monotonic solutions and solutions which oscillate an infinite number of times and tend to a certain limit. The properties of solutions depend on the suction parameter. Furthermore, making use of the fourth-order Runge-Kutta scheme together with the shooting method, numerical solutions are obtained.

keywords: Boundary layer, Falkner-Skan, Similarity solution, Pseudo-similarity.
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1. Introduction

In this paper we are concerned with the classical two-dimensional laminar incompressible boundary layer flow past a wedge or a flat plate [29]. For the first approximation, the model is described by the Prandlt equations or the boundary layer equations

$$u\partial_x u + v\partial_y u = U_e\partial_x U_e + \nu\partial^2_{yy}u, \quad \partial_x u + \partial_y v = 0, \tag{1.1}$$

where (x, y) denote the usual orthogonal Cartesian coordinates parallel and normal to the boundary $y = 0$ (the wall),$u(x, y), v(x, y)$ are the x and y velocity components, respectively, and $\nu > 0$ is the kinematic viscosity. The function $U_e = U_e(x)$ is a given external velocity flow (the main-stream velocity) which is assumed throughout the paper to be nonnegative and is such that $u(x, y)$ tends to $U_e(x)$ as $y \to \infty$. Equations (1.1) can be written in the form

$$\partial_y\psi\partial^2_{xy}\psi - \partial_x\psi\partial^2_{yy}\psi = U_e\partial_x U_e + \nu\partial^3_{yyy}\psi, \tag{1.2}$$

where ψ is the well–known stream function defined by $u = \partial_y\psi, v = -\partial_x\psi$.
This equation with appropriate external velocity flow has been the main focus of studies of particular exact solutions. It is well known that to derive properties of solutions to a nonlinear partial differential equations we use a family of special solutions. They play an important role for describing the intermediate asymptotic behavior of classes of solutions of original problems with arbitrary initial data (see for example [4],[30]). A crucial step, in the analysis, is to get favorable conditions such that particular solutions exist. For the boundary layer problems, research on this subject has a long history, which dates to the pioneering works by Blasius [6] and Falkner and Skan [13]. Their investigations lead to solutions to (1.2) in the form

$$\psi(x, y) = x^\alpha f(yx^{-\beta}). \tag{1.3}$$

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Therefore, if ψ is a such solution, it is easily verified that, for $\beta \neq 0$,

$$\psi(x,y) = \lambda^{-\frac{\alpha}{\beta}} \psi(\lambda^{\frac{1}{\beta}} x, \lambda y),$$

for all $\lambda > 0$, and we immediately see that

$$\psi(x,y) = \left(\frac{x}{x_0}\right)^{\alpha} \psi\left(x_0, y\left(\frac{x}{x_0}\right)^{-\beta}\right).$$

This means that a solution $\psi(x,y)$ for y fixed is similar to the solution $\psi(x_0,y)$ at a certain x_0. This solution is called invariant or similarity solution and the function f is called the shape function or the dimensionless stream function.
The main goal of identifying similarity solutions is to reduce the original problem to an ordinary differential equation which is easier to analyze. One says that the function (1.3) scales the partial differential equation (1.2) if the function f satisfies an ordinary differential equation called similarity boundary layer equation. We refer the reader to [2],[7],[8], [21], [22], [25] and the references therein. A relation of the type $\alpha = h(\beta)$ called scaling relation.
In [6] the Blasius equation

$$f''' + \frac{1}{2} f f'' = 0 \quad \text{on} \quad (0,\infty) \tag{1.4}$$

is derived from (1.1) and (1.3) where U_e is assumed to be a constant function. In the above equation the primes indicate differentiations with respect to the similarity variable $t = yx^{-1/2}$. Here we have $\alpha = \beta = \frac{1}{2}$. The Blasius equation is also a particular case of the Falkner-Skan equation [13]

$$f''' + \frac{m+1}{2} f f'' = m(f'^2 - 1) \quad \text{on} \quad (0,\infty), \tag{1.5}$$

where the external velocity $U_e(x) = U_\infty x^m, (U_\infty > 0)$. The exponents α, β are given by $\alpha = \frac{m+1}{2}, \beta = \frac{1-m}{2}$.
Guided by the results of [6], [13] attention will be given, in Section 2, to identify a class of external velocities for equation (1.2) to possess solutions under the form (1.3) where $\alpha + \beta = 1$. Additionally, we investigate, in Section 3, the similarity solutions to (1.2), where both of the external velocity and the stretching velocity of the permeable surface are assumed to vary as x^{-1}.

2. Similarity solutions

As it is said in the introduction, this work deals with the similarity steady boundary–layer flow induced by an incompressible viscous fluid past a semi–infinite flat plate. The phenomenon is governed by the system

$$u\partial_x u + v\partial_y u = \nu \partial^2_{yy} u + U_e \partial_x U_e, \quad \partial_x u + \partial_y v = 0. \tag{2.1}$$

accompanied by the boundary condition

$$u(x,y) \to U_e(x) \quad \text{as } y \to \infty. \tag{2.2}$$

Therefore the stream function ψ satisfies

$$\partial_y \psi \partial^2_{xy} \psi - \partial_x \psi \partial^2_{yy} \psi = U_e \partial_x U_e + \nu \partial^3_{yyy} \psi, \tag{2.3}$$

and

$$\lim_{y\to\infty} \partial_y \psi(x,y) = \partial_y \psi(x,\infty) = U_e(x). \tag{2.4}$$

The main problems, arising in the study of similarity solutions, are related to the existence of the exponents α and β and to the rigorous study of the differential equation satisfied by the profile f, which is, in general, nonlinear. For equation (2.3), the classical approach for identifying α and β is the scaling and transformation

group [4]. The essential idea is to seek a and b such that if ψ satisfies (2.3) the new function $\psi_\lambda(x,y) = \lambda^a\psi(\lambda^b x, \lambda y)$ is also a solution. This will be certainly possible if the external velocity field U_e is subject to the transformation. The parameters a and b may depend on α and β. Using (2.3), it is easily verified that ψ_λ is a solution, for any λ, if the following

$$a + b = 1, \quad \lambda^{3+a} U_e(\lambda^b x) U_{e_x}(\lambda^b x) = U_e(x) U_{e_x}(x), \tag{2.5}$$

hold for any x (see the proof of Theorem 2.1 below). In [13] Falkner and Skan considered the case where

$$U_e(x) = U_\infty x^m, \tag{2.6}$$

where $U_\infty (> 0)$ and m are constants. The case $m = 0$ was treated earlier by Blasius [6]. Therefore, we deduce from (2.5), for $m \neq 1$,

$$a = \frac{m+1}{m-1}, \quad b = -\frac{2}{m-1}.$$

Now assume that ψ satisfies the invariance property $\psi_\lambda = \psi$. Thus if we set $\lambda^b x = 1$, we find that ψ can be written in the form

$$\psi(x,y) = x^\alpha f(yx^{-\beta}),$$

where

$$\alpha = -\frac{a}{b}, \beta = \frac{1}{b}.$$

This leads to (1.3) with $\alpha = \frac{m+1}{2}$ and $\beta = -\frac{m-1}{2}$. This, may be, helps us to understand the similarity stream functions obtained by Blasius and Falkner and Skan. These authors considered solutions to (2.3),(2.4) of the form

$$\psi = (\nu U_e x)^{\frac{1}{2}} f(t), \quad t = (U_e/\nu x)^{\frac{1}{2}} y. \tag{2.7}$$

In fact, we shall see in the following results, that condition (2.6) is necessary for problem (2.3),(2.4) to admit similarity solutions under the form (1.3). The main result of this section says that (1.3) scales (2.3) if the external velocity satisfies

$$U_e(x) = U_\infty (x + x_0)^m,$$

for some x_0, where $m = \alpha - \beta$. We note, in passing that if ψ is a solution the function $\overline{\psi}(x,y) = \psi(x + x_0, y)$ satisfies (2.3) with $U_e(x + x_0)$ instead of $U_e(x)$. Then it allows x_0 to be zero. The necessary condition on U_e can now be stated as follows.

THEOREM 2.1. *Assume that equation* (2.3) *has a similarity solution in the form* (1.3) *where* $\alpha + \beta = 1$. *Then, there exist two nonnegative constants,* c_1, c_2 *such that*

$$U_e^2(x) = c_1 x^{2m} + c_2, \tag{2.8}$$

for all $x > 0$, *where* $m = \alpha - \beta$.

PROOF. Let ψ be a stream function to (2.3) defined by (1.3) where $\alpha + \beta = 1$. Assume first that $\beta \neq 0$. We choose $a = -\frac{\alpha}{\beta}$, $b = \frac{1}{\beta}$, and define $\psi_\lambda(x,y) = \lambda^a \psi\left(\lambda^b x, \lambda y\right)$. Hence $a + b = 1, \psi \equiv \psi_\lambda$ and

$$L(\psi_\lambda)(x,y) = \lambda^{a+3} L(\psi)(\lambda^b x, \lambda y)$$

for any $\lambda > 0$, where L is the operator defined by

$$L(\psi) = \partial_y\psi\partial^2_{xy}\psi - \partial_x\psi\partial^2_{yy}\psi - \nu\partial^3_{yyy}\psi.$$

According to equation (2.3) we deduce for any $\lambda > 0$ and any $x > 0$,

$$h(x) = \lambda^{a+3} h(\lambda^b x),$$

where $h(x) = U_e(x)\partial_x U(x)$. In particular, for fixed $x_0 > 0$

$$h(\lambda^b x_0) = \lambda^{-(a+3)} h(x_0).$$

Setting $x = \lambda^b x_0$ we infer

$$h(x) = x^{-\frac{a+3}{b}} x_0^{\frac{a+3}{b}} h(x_0).$$

Solving the equation

$$U_e \frac{dU_e}{dx} = x^{-\frac{a+3}{b}} x_0^{\frac{a+3}{b}} h(x_0) \tag{2.9}$$

yields us (2.8) for $\beta \neq 0$, since $-\frac{a+3}{b} + 1 = 2(\alpha - \beta)$.
For $\beta = 0$, hence $\alpha = m = 1$, the new function

$$\psi_\lambda(x, y) = \lambda^a \psi(\lambda^{-a} x, y),$$

for any fixed $a \neq 0$, is equivalent to ψ and satisfies

$$L(\psi_\lambda)(x, y) = \lambda^a L(\psi)(\lambda^{-a} x, y)$$

for any $\lambda > 0$. Arguing as in the case $\beta \neq 0$ one arrives at (2.8) with $m = 1$. ∎

A similarity assumption to (2.9) was proposed by Spalding [31] and studied extensively by Evans [12]. The authors assumed that U_e satisfies the equation

$$\frac{dU_e}{dx} = C U_e^{2(\gamma-1)/\gamma}, \tag{2.10}$$

where C is a constant. The similarity equation is then

$$f''' + ff'' + \gamma(1 - f'^2) = 0, \tag{2.11}$$

which is equivalent to (1.5) with $\gamma = \frac{2m}{m+1}$. Now let us discuss the consequence of the boundary condition at infinity. We note, in passing, that from (2.8) and (2.10) we easily deduce that $U_e(x) = U_\infty x^m$.

Theorem 2.2. *Equation* (2.3) *has a similarity solution,* ψ*, in the form* (1.3) *such that* $\alpha+\beta = 1$ *and* $\partial_y \psi(., \infty) = U_e$*, if and only if there exists a real* m *such that Then*

$$U_e(x) = U_\infty x^m, \tag{2.12}$$

for all $x > 0$*,* U_∞ *is a constant. Moreover the real* m *satisfies* $m = \alpha - \beta$.

Proof. Let $U_e(x) = U_\infty x^m$. the existence of similarity solutions in the form (1.3) where $\alpha = \frac{m+1}{2}$ and $\beta = \frac{m-1}{2}$ is given in [6] and [13] (see also [17],[18]) for some values of m. Conversely, assume that (2.3), (2.4) has a similarity solution in the form (1.3) with $\alpha + \beta = 1$. Then $U_e(x)^2 = c_1 x^{2m} + c_2$ and $\lim_{y\to\infty} x^{2m} (f'(yx^{-\beta})^2 = c_1 x^{2m} + c_2$. Thereafter, the function f'^2 has a finite limit at infinity, which is unique and is given by $c_1 + c_2 x^{-2m}$. This is acceptable only for $c_2 = 0$. ∎

Remark 2.1. It may be noted that in the above theorem it is not required to find the range of the real m such that problem (2.3),(2.4) has a similarity solution in the form (1.3). In fact, the similarity equation may have no solution for some real m (see Section 3). Theorem 2.2 indicates, in particular, that for a prescribed external velocity $U_e(x) = U_\infty x^m$, the reals α and β such that (1.3) scales (2.3) are given by

$$\alpha = \frac{m+1}{2}, \beta = -\frac{m-1}{2}.$$

REMARK 2.2. For a general external velocity, our approach can be used to obtain particular solutions. In [25] it is obtained particular solutions, having the form

$$\psi = H(x,y)f(\eta(x,y)) + \psi_0.$$

So, if

$$U_e(x) = Ax^m + \mathcal{U}(x)$$

we conjecture that there exist solutions in the form

$$\psi(x,y) = x^{\alpha} f(yx^{-\beta}) + \psi_0(x,y),$$

where $\alpha = \frac{m+1}{2}$ and $\beta = -\frac{m-1}{2}$. The function ψ_0 may be similarity and connected to $x^{\alpha} f(yx^{-\beta})$ and $\mathcal{U}(x)$. In the next result we shall identify the external velocity such that equation (2.3) has a solution in the form

$$\psi(x,y) = x^{\alpha} f(yx^{-\beta}) + cx^n y, \ \alpha + \beta = 1, \quad c = const. \tag{2.13}$$

Note that the function $\psi_0(x,y) = cx^n y$ can be written in the form $\psi_0(x,y) = cx^{\alpha_1}(yx^{-\beta_1})$, for any α_1, β_1 such that $\alpha_1 - \beta_1 = n$; that is ψ_0 is similarity. So, for $\alpha - \beta = n$ (2.13) is identically to (1.3). Therefore we assume that $\alpha - \beta \neq n$.

THEOREM 2.3. *Let α, β are reals such that $\alpha - \beta \neq n$. The reals α, β in* (2.13) *scale* (2.3) *if and only if $\alpha = \frac{2}{3}, \beta = \frac{1}{3},\ n = -\frac{1}{3}$ and the following*

$$U_e(x) = Ax^{\frac{1}{3}} + cx^{-\frac{1}{3}},$$

holds for some constant A.

PROOF. For $U_e(x) = ax^{1/3} + cx^{-1/3}$, it is shown in [25] that the function $\psi(x,y) = x^{2/3} f(yx^{-1/3}) + cyx^{-1/3}$ satisfies (2.3)
Now, assume that (2.3) has a solution in the form (2.13). Substituting (2.13) into (2.3) yields

$$\left\{ \begin{array}{l} x^{2\alpha-2\beta-1}\Big[\nu f''' \ + \ \alpha f f'' \quad -(\alpha-\beta)f'^2\Big] \ + \ c(n+\alpha-\beta)x^{n+\alpha-\beta-1}f' \\ \qquad\qquad +nc(n+\beta)tx^{n+\alpha-\beta-1}f'' - cnx^{2n-1} + U_e\frac{d\ U_e(x)}{dx} = 0, \end{array} \right. \tag{2.14}$$

where $t = yx^{-\beta}$. Since $\alpha - \beta \neq n$, the function f satisfies an ordinary differential equation if and only if

$$n + \alpha - \beta = n + \beta = 0, \tag{2.15}$$

and there exists a real γ such that

$$\left\{ \begin{array}{lcl} \nu f''' \ + \ \alpha f f'' - (\alpha-\beta)f'^2 & = & \gamma, \\ -cnx^{2n-1} + U_e \partial_x U_e & = & \gamma x^{2(\alpha-\beta)-1}. \end{array} \right. \tag{2.16}$$

From (2.13) and $(2.14)_2$ we deduce immediately that $\alpha = \frac{2}{3}, \beta = \frac{1}{3},\ n = -\frac{1}{3}$ and $U_e(x) = Ax^{\frac{1}{3}} + cx^{-\frac{1}{3}}$. ■

REMARK 2.3. Let us now derive the well known Blasius and Falkner-Skan. Of course the external velocity is given by $U_e(x) = U_\infty x^m$. We recall that for the Blasius model we have $m = 0$ and the case $m \neq 0$ was considered by Falkner–Skan. Instead of taking $\alpha = \frac{m+1}{2}$ and $\beta = -\frac{m-1}{2}$ we shall insert (1.3) into (2.3) and choose α and β such that f satisfies an ordinary differential equation. Obviously we shall obtain that α and β must to be , respectively $\frac{m+1}{2}$ and $-\frac{m-1}{2}$.
Problem (2.3),(2.4) is written as

$$\partial_y \psi \partial^2_{xy}\psi - \partial_x \psi \partial^2_{yy}\psi = \nu \partial^3_{yyy}\psi + mU^2_\infty x^{2m-1}, \tag{2.17}$$

with the boundary conditions

$$\partial_x\psi(x,0) = \partial_y\psi(x,0) = 0, \quad \partial_y\psi(x,\infty) = U_\infty x^m. \tag{2.18}$$

If we substitute (1.3) into the first equation of this problem we obtain

$$(\alpha-\beta)x^{2(\alpha-\beta)-1}(f')^2 - \alpha x^{2(\alpha-\beta)-1}ff'' - x^{\alpha-3\beta}f''' - mU_\infty^2 x^{2m-1} = 0,$$

and this is an ordinary differential equation if and only if

$$2(\alpha-\beta)-1 = \alpha-3\beta = 2m-1,$$

(the scaling relation) i.e.

$$\alpha = \frac{m+1}{2} \quad \text{and} \quad \beta = -\frac{m-1}{2}.$$

After a scaling ($\sqrt{\nu U_\infty} f\left(.\sqrt{\frac{U_\infty}{\nu}}\right)$ instead of $f(.)$) the corresponding ordinary differential equation is

$$f''' + \frac{m+1}{2}ff'' + m(1-f'^2) = 0. \tag{2.19}$$

The prime denotes the derivative with respect to $t = \sqrt{\frac{U_\infty}{\nu}}yx^{\frac{m-1}{2}}$. The boundary conditions read

$$f(0) = f'(0) = 0, f'(\infty) = 1. \tag{2.20}$$

Remark 2.4. We observe, in passing, that the boundary conditions on the plate are not required. This means that our analysis works even if we have a continuous stretching surface. In this case, if the stretching velocity is given by $U_s(x) = U_w x^n$, we deduce from (1.3), $n = \alpha - \beta$ which is the same exponent as that of the external velocity. If $v = 0$ at $y = 0$ the function f satisfies [17],[18]

$$f''' + \frac{m+1}{2}ff'' - m(f'^2 - 1) = 0,$$

$$f(0) = 0, \ f'(0) = \zeta, \ f'(0) = 1,$$

where ζ is equal to the ratio of the free stream velocity to the boundary velocity.
In the case where the external velocity is zero, the stretching and the suction/injection velocity have the form

$$u(x,0) = U_w x^m, \quad v(x,0) = -V_w x^{\frac{m-1}{2}},$$

the similarity equation is given by

$$\begin{cases} f''' + \frac{m+1}{2}ff'' - mf'^2 = 0, \\ \\ f(0) = a, \ f'(0) = 1, \ f'(\infty) = 0, \end{cases} \tag{2.21}$$

where a is a real (the suction/injection parameter). This problem arises also in the study the free convection, along a vertical flat plate embedded in a porous medium. We refer the reader to the papers [26],[8],[5],[16] and the references therein for a complete physical derivation and analysis of this problem.

Remark 2.5. We finally, mention a result of [30] which provides that for the external velocity satisfying (2.12) where $U_\infty > 0$ and $m > 0$, any solution u to (1.1) with $U_w = V_w = 0$ such that u_y is continuous for $0 < x < \infty, 0 \le y < \infty$, satisfies the fundamental asymptotic estimate

$$\left|\frac{u}{U_e} - f'\right| = o\left(\frac{1+m\log x}{x^m}\right),$$

as $x \to \infty$, uniformly in y, where f is a solution to (2.19),(2.20).

3. The pseudo-similarity solutions

In the present section we focuss our attention to the case $m = -1$. The case $m \neq -1$ has been abundantly studied. In particular it was considered for the Falkner Skan equation (2.11) where $\gamma = 2m/(m+1)$. The cases $\gamma = 0$ and $\gamma = 1/2$ are referred as Blasius and Homann differential equations, respectively. In [11] Coppel classified all solutions of (2.11), where $0 \leq \gamma < 2$. Craven and Peletier [10] proved that equation (2.11), where $\frac{1}{2} < \gamma \leq 1$, has at most one solution f satisfying the boundary condition

$$f(0) = f'(0) = 0, \ f'(\infty) = 1. \tag{3.1}$$

In [19] Hasting and Siegel showed that for $|\gamma|$ sufficiently small there exists a unique solution to (2.11),(3.1) satisfying $|f'| < 1$ and $f''(0) < 0$. The case $|\gamma| > 1$ can be found in the works [17, 18] by Hasting and Troy. In [17] deals with $\gamma > 1$. In this it is shown that the equation has a periodic solution and for any integer N problem (2.11),(3.1) has a solution with at least N relative minima. It is also mentioned that as γ increases, the structure of periodic solutions and solutions to (2.11),(3.1) gets progressively more complicated.
In the present section we restrict our attention to the case $m = -1$. Therefore we shall consider the external velocity is given by $U_e(x) = \frac{U_\infty}{x}(m = -1)$, and get new solutions to the problem

$$\partial_y \psi \partial^2_{xy} \psi - \partial_x \psi \partial^2_{yy} \psi = \nu \partial^3_{yyy} \psi - U_\infty x^{-3}, \tag{3.2}$$

subject to the boundary conditions

$$\partial_y \psi(x,0) = U_w.x^{-1}, \quad \partial_x \psi(x,0) = -V_w x^{-1}, \quad \partial_y \psi(x,\infty) = U_\infty.x^{-1}, \tag{3.3}$$

where V_w is a real and U_w and U_∞ are nonnegative and satisfy $U_w < U_\infty$. The subject of the present section is motivated by the work of Magyari, Pop and Keller [27],where the case $U_\infty = 0$ was considered and discussed in detail. The authors showed that if $m = -1$ problem (2.21) has no solution. In order to overcome this difficulty the authors showed that the term $V_w log(x)$ must be added to expression (1.3). It has been also confirmed, by numerical calculations, that new solutions (pseudo-similarity solutions) exist provided that the suction parameter is large.
For (3.2),(3.3), where $U_\infty \neq 0$ and according to Section 2, the function ψ can be written as

$$\psi(x,y) = \sqrt{\nu U_\infty} f(\sqrt{\frac{U_\infty}{\nu}} y x^{-1}).$$

Since, in general $V_w = \frac{m+1}{2}\sqrt{\nu U_\infty} f(0)$, we deduce $V_w = 0$ for $m = -1$ and the function $\theta = f'$ satisfies

$$\begin{cases} \theta'' + \theta^2 - 1 = 0, \\ \\ \theta(0) = \zeta, \theta(\infty) = 1, \end{cases} \tag{3.4}$$

where $\zeta = \frac{U_w}{U_\infty}$ is in the interval $[0, 1)$.
The stability of equilibrium point $(1, 0)$ of (3.4) cannot be determined from the linearization. To analyze the behavior of the nonlinear equation $(3.4)_1$, we observe that

$$E'(t) = 0,$$

where E is the Lyapunov function defined by

$$E(t) = \frac{1}{2}\theta'(t)^2 + \frac{1}{3}\theta(t)^3 - \theta(t).$$

Then, for some constant C, the following

$$\theta' = \pm\sqrt{2}\left(C + \theta - \frac{1}{3}\theta^3\right)^{1/2},$$

holds. The analysis of the algebraic equation of the phase path in the phase plane (θ, θ') reveals that the equilibrium point $(1, 0)$ is a center. Hence Problem (3.4) has no solution for any $\zeta > -1$ except the trivial one $\theta = 1$, (see *Fig.*3.1).

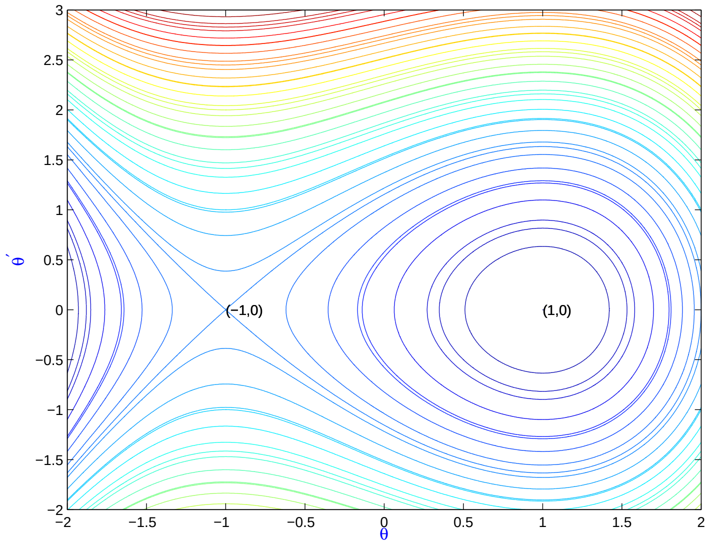


Fig. 3.1 Classification of solutions of $\theta'' + \theta^2 - 1 = 0$ according to $\theta(0)$ and $\theta'(0)$.

Remark 3.1. If we impose the condition $\theta(\infty) = -1$ instead of $\theta(\infty) = 1$–which is also of physical interest– it is easy to see that for any $\zeta \leq 2$ there exists a unique solution–up to translation–. This solution satisfies

$$\frac{1}{2}\theta'(t)^2 + \frac{1}{3}\theta(t)^3 - \theta(t) = \frac{2}{3},$$

and we find that

$$\theta(t) = 2 - 3\tanh^2\left[\pm t/\sqrt{2} + \operatorname{arctanh}\left\{(2-\zeta)/3)^{1/2}\right\}\right].$$

To obtain exact solutions to (3.2),(3.3), we introduce the following Anzats for the stream-function

$$\psi(x,y) = ax^\alpha F(x, byx^{-\beta}). \tag{3.5}$$

Guided by the analysis of Section 2 we take $\alpha = \frac{m+1}{2}, \beta = -\frac{m-1}{2}, a = \sqrt{\nu U_\infty}$ and $b = \frac{U_\infty}{\nu}$. Here the real m is assumed to be any real. Hence boundary condition (3.3) reads

$$\partial_y\psi(x,0) = U_w x^m, \quad \partial_x\psi(x,0) = -V_w x^{\frac{m-1}{2}}, \quad \partial_y\psi(x,\infty) = U_\infty x^m.$$

Using this and the equation of the stream function we find that F satisfies

$$\begin{cases} F''' + \frac{1+m}{2}FF'' - m(F'^2 - 1) + x\left(F''\partial_x F - F'\partial_x F'\right) = 0, \\ F'(x,0) = \zeta, \quad F'(x,\infty) = 1, \\ \dfrac{1+m}{2}F(x,0) + x\partial F(x,0) = \dfrac{V_w}{\sqrt{\nu U_\infty}}, \end{cases} \tag{3.6}$$

where the primes denote partial differential with respect to $t = \sqrt{\frac{U_\infty}{\nu}}yx^{\frac{m-1}{2}}$. By writing

$$F(x,t) = f(t) + H(x),$$

we find

$$\begin{cases} f''' + \frac{1+m}{2}ff'' - m\left(f'^2 - 1\right) + f''\left(xH' + \frac{1+m}{2}H\right) = 0, \\ \dfrac{1+m}{2}f(0) + \dfrac{1+m}{2}H(x) + xH'(x) = \dfrac{V_w}{\sqrt{\nu U_\infty}}, \quad f'(0) = \zeta \in [0,1), \quad f'(\infty) = 1. \end{cases} \tag{3.7}$$

Hence, there exists a real τ such that

$$(3.8)\qquad \begin{cases} f''' + \frac{1+m}{2} f f'' - m\left(f'^2 - 1\right) + \tau f'' = 0, \quad t > 0, \\ xH' + \frac{1+m}{2} H = \tau, \quad x > 0, \\ \dfrac{1+m}{2} f(0) + \tau = \dfrac{V_w}{\sqrt{\nu U_\infty}}, \quad f'(0) = \zeta, \quad f'(\infty) = 1. \end{cases}$$

First, let us note that (2.19) and $(3.8)_1$ are equivalent for $m \neq -1$. In this case the general solution of $(3.8)_2$ is

$$H(x) = Cx^{-\frac{1+m}{2}} + \frac{2\tau}{1+m},$$

where C is a constant. Thus the stream function is given by $\psi(x,y) = aC + ax^{\frac{1+m}{2}}\left(f(t) + \frac{2\tau}{1+m}\right)$, and the new function $g = f + \frac{2\tau}{1+m}$ satisfies equation (2.19).
Thereafter, we will assume that $m = -1$ and this leads to

$$(3.9)\qquad \begin{cases} f''' + \tau f'' + f'^2 - 1 = 0, \\ f'(0) = \zeta, \quad f'(\infty) = 1, \end{cases}$$

and

$$H(x) = \tau \log x + C, \quad C = const.$$

Then the required exact solution has the form

$$(3.10)\qquad \psi(x,y) = af(byx^{-1}) + a\tau \log(x).$$

This formula is similar to the one studied in [24] in the context of rough surface growth and can be regarded as solution with dynamic scaling [24]. In passing, we note that from (3.8) one sees $\tau = V_w \left(\nu U_\infty\right)^{-1/2}$. This means that the constant τ plays the role of suction/injection parameter.
To study (3.9) it is more convenience to analysis the second ordinary differential equation

$$(3.11)\qquad \begin{cases} \theta'' + \tau\theta' + \theta^2 - 1 = 0, \\ \theta(0) = \zeta, \quad \theta(\infty) = 1, \end{cases}$$

where $0 \le \zeta < 1$ and $\tau \neq 0$. In fact the real τ will be taken in $(0, \infty)$. The existence of solutions to (3.11) will be proved by means of shooting method. Hence, the boundary condition at infinity is replaced by the condition $\theta'(0) = d$, where d is a real. For any d the new initial–value problem has a unique local solution θ_d defined in its maximal interval of existence $(0, T_d), T_d \le \infty$. We shall see that for an appropriate d the solution θ_d is global and satisfies

$$(3.12)\qquad \theta_d(\infty) = 1.$$

A simple analysis in the phase plane reveals that problem (3.11) may have solutions only for $\tau > 0$. In fact the ordinary differential equation in (3.11) is considered as a nonlinear autonomous system in $\mathbb{R}^2$, with the unknown (θ, θ'), mainly

$$(3.13)\qquad \begin{cases} \theta' = \varphi, \\ \varphi' = -\tau\varphi + 1 - \theta^2, \end{cases}$$

subject to the boundary condition

$$(3.14)\qquad \theta(0) = \zeta, \quad \varphi(0) = d.$$

The linear part of the above system at the equilibrium point $(1,0)$ is defined by the matrix

$$J = \begin{pmatrix} 0 & 1 \\ -2 & -\tau \end{pmatrix}.$$

The eigenvalues of J are

$$\lambda_1 = \frac{-\tau - \sqrt{\tau^2 - 8}}{2}, \quad \lambda_2 = \frac{-\tau + \sqrt{\tau^2 - 8}}{2},$$

if $\tau \geq \sqrt{8}$ and for $0 < \tau < \sqrt{8}$,

$$\lambda_1 = \frac{-\tau - i\sqrt{8 - \tau^2}}{2}, \quad \lambda_2 = \frac{-\tau + i\sqrt{8 - \tau^2}}{2}.$$

Therefore, the hyperbolic equilibrium point $(1,0)$ is asymptotically stable if τ is positive and unstable for τ is negative. In particular problem (3.11) has no nontrivial solutions if $\tau < 0$. If $\tau > 0$ we deduce from the above that there exists $\delta > 0$ such that for any d and ζ such that $d^2 + (\zeta - 1)^2 < \delta^2$ the local solution θ_d is global and satisfies (3.12). In the following we construct solutions to (3.11) where the condition $d^2 + (\zeta - 1)^2 < \delta^2$ is not necessarily required. For a mathematical consideration the parameter ζ will be taken in $(-1, \sqrt{3}]$. The following theorem deals with nonnegative values of ζ.

THEOREM 3.1. *Let $0 \leq \zeta \leq \sqrt{3}$ and d be a real such that*

$$d^2 \leq 2\zeta(1 - \frac{\zeta^2}{3}). \tag{3.15}$$

Then the local solution θ_d is global, nonnegative and tends to 1 as t approaches infinity.

The key point of the proof of this theorem is to find d such that $\theta_d(t) > 0$ for all $t > 0$. To show this we consider again the function $E(\theta(t), \varphi(t)) = \frac{1}{2}\varphi(t)^2 + \frac{1}{3}\theta^3 - \theta$. Along an orbit we have

$$\frac{d}{dt} E(\theta(t), \varphi(t)) = -\tau\varphi(t)^2 \leq 0.$$

Hence

$$E(\theta_d(t), \theta_d'(t)) \leq E(\zeta, d),$$

as long as $\theta_d(t)$ exists; that is $t < T_d$. The following result shows that $\theta_d \geq 0$ on $(0, T_d)$ and then $T_d = \infty$.

LEMMA 3.1. *Let $0 \leq \zeta \leq \sqrt{3}$ and d satisfying (3.15). Then θ_d is nonnegative, global, bounded and its first derivative also is bounded.*

PROOF. First we note that from (3.13) and (3.14), there exists $t_0 > 0$, small, such that θ_d is positive on $(0, t_0)$. Assume that θ_d vanishes at some $t_1 > t_0$ and suppose that $\theta_d'(t_1) \neq 0$. Because

$$E(\zeta, d) \geq E(\theta_d(t), \theta_d'(t)) \geq \frac{1}{2}\theta_d'(t_1)^2,$$

for all $0 \leq t \leq t_1$. we deduce $\frac{1}{2}d^2 > \zeta(1 - \frac{1}{3}\zeta^2)$, which contradicts (3.15). Therefore $\theta_d'(t_1) = 0$. In this case we deduce from the equation of θ_d that $\theta_d''(t_1) = 1$ and then θ_d is nonnegative on a some neighborhood of t_1. Consequently the local solution is nonnegative as long as there exists. To show that θ_d is global we note that

$$E(\zeta, d) \geq \frac{1}{2}\theta_d'(t)^2 + \frac{1}{3}\theta_d^3(t) - \theta_d(t) \geq -\frac{2}{3},$$

for all $t \leq T_d$, since θ_d is nonnegative. Hence θ_d and (then) θ_d' are bounded. Consequently θ_d is global. The lemma is proved ■

Now, we are ready to prove Theorem 3.1. More precisely we have.

Proposition 3.1. *Let $\tau > 0, \zeta \in [0, \sqrt{3}]$ and d such that* (3.15) *holds. Then, the global solution θ_d to* (3.13),(3.14) *satisfies the boundary condition* (3.12) *at infinity.*

On account of Lemma 3.1 we need only to show that θ_d goes to 1 as t approaches infinity. Since this result is the broad goal of the present section, we give two proofs.

The first proof.

Using the fact that θ_d and θ'_d are bounded we deduce from the equation of θ_d that θ''_d is bounded. On the other hand, since $\frac{d}{dt}E(\theta_d, \theta'_d) = -\tau {\theta'_d}^2$, the function θ'_d is square integrable. Now, we use the identity

$$\theta'_d(t)^3 = \theta'_d(0)^3 + 3\int_0^t \theta'_d(s)^2 \theta''_d(s)ds,$$

to show that $\theta'_d(t)$ has a finite limit as t tends to infinity and this limit is zero. Next, we get, by differentiation $(3.11)_1$

$$\theta'''_d = -\tau\theta'' - 2\theta_d\theta'_d. \tag{3.16}$$

Multiplying equation (3.16) by θ''_d, integrating the equation obtained over $(0, t)$, we get

$$\tau\int_0^t \theta''_d(s)^2 ds = \frac{1}{2}\theta''_d(0)^2 + \theta_d(0)\theta'_d(0)^2 - \frac{1}{2}\theta''_d(t)^2 - \theta_d(t)\theta'_d(t)^2 + \int_0^t \theta'_d(s)^3 ds.$$

This implies $\theta''_d \in L^2(0, \infty)$. Since θ'''_d is bounded, by using (3.16), one sees $\lim_{t\to\infty} \theta''_d(t) = 0$. Finally, we deduce from $(3.11)_1$ that $\theta_d(t)$ goes to 1 as t tends to infinity since θ_d is nonnegative.

The second proof.

This proof uses the Bendixson Criterion. Let $\mathcal{T}$ be the trajectory of (θ_d, θ'_d) in the phase plane $(0, \infty) \times \mathbb{R}$ for $t \geq 0$ and let $\Gamma^+(\mathcal{T})$ be its w-limit set at $+\infty$. From the boudedness of $\mathcal{T}$ it follows that $\Gamma^+(\mathcal{T})$ is a nonempty connected and compact subset of $(0, \infty) \times \mathbb{R}$ (see, for example [1, p 226]). Moreover $(-1, 0) \notin \Gamma^+(\mathcal{T})$, since θ_d is nonnegative. Note that if $\Gamma^+(\mathcal{T})$ contains the equilibrium point $(1, 0)$ then $\Gamma^+(\mathcal{T}) = \{(1, 0)\}$, since $(1, 0)$ is asymptotically stable. Assume that $(1, 0) \notin \Gamma^+(\mathcal{T})$, Applying the Poincaré–Bendixson Theorem [14, p 44] we deduce that $\Gamma^+(\mathcal{T})$ is a cycle, surrounding $(1, 0)$. To finish the second proof we shall prove the nonexistence of such a cycle. To this end we define $P(\theta, \varphi) = \varphi, Q(\theta, \varphi) = -\tau\varphi + 1 - \theta^2, \varphi = \theta'_d$ and $\theta = \theta_d$. The function (θ, φ) satisfies the system $\theta' = P(\theta, \varphi)$, $\varphi' = Q(\theta, \varphi)$. Let D be the bounded domain of the (θ, φ)–plane with boundary Γ^+. As P and Q are regular we deduce, via the Green–Riemann theorem,

$$\int\int_D (\partial_\varphi Q + \partial_\theta P)\, d\varphi d\theta = \int_{\Gamma^+} (Q d\theta - P d\varphi) = 0, \tag{3.17}$$

thanks to the system satisfied by (θ, φ). But $\partial_\varphi Q + \partial_\theta P = \tau$ which is positive. We get a contradiction. ■

In the same way as in the proof of Proposition 3.1, we can see that any global solution to (3.13) which is bounded from bellow by some $a > -1$ tends to 1 at infinity. Therefore, to complete our analysis, we shall determine a domain of attraction of the critical point $(1, 0)$. Let

$$\mathcal{P} = \left\{(\zeta, d)\ \in \mathbb{R}^2 : \zeta > -1, \frac{1}{2}d^2 + \zeta\left(\frac{1}{3}\zeta^2 - 1\right) < \frac{2}{3}\right\}.$$

Proposition 3.2. *For any (d, ζ) in $\mathcal{P}$ the local solution to (3.13),(3.14) is global and converges to 1 at infinity.*

Proof. Let us consider a one-parameter of family of curves defined by

$$E(\theta, \varphi) = \frac{1}{2}\varphi^2 + \frac{1}{3}\theta^3 - \theta = C,$$

where C is a real parameter. Note that, in the phase plane, this family is solution curves of system (3.4). The curve $\varphi^2 = 2\theta - \frac{2}{3}\theta^3 + \frac{4}{3}$, corresponding to $C = \frac{2}{3}$, goes through the point $(2, 0)$ and has the saddle $(-1, 0)$

($\tau = 0$) as its α and w-limit sets, see Fig 3.2. We denote this solution curve by $\mathcal{H}$, which is, in fact, an homoclinic orbit and define a separatrix cycle for (3.4). We shall see that the bounded open domain with the boundary $\mathcal{H}$ is an attractor set for $(1,0)$ of system (3.13) where $\tau > 0$. This domain is given by $E(\theta, \varphi) = C, \theta > -1$, for all $-\frac{2}{3} \leq C < \frac{2}{3}$, which is $\mathcal{P}$. As $\frac{d}{dt}E \leq 0$ any solution, with initial data in $\mathcal{P}$ cannot leave $\mathcal{P}$, (see the proof of Theorem 3.1). By LaSalle invariance principle we deduce that for any (ζ, d) in $\mathcal{P}$ the w-limit set, $\Gamma^+(\zeta, d)$ is a nonempty, connected subset of $\mathcal{P} \cap \{\varphi = 0\}$, (see [1, p 234].) However, if $\theta \neq 1, \varphi = 0$ is a transversal of the phase–flow, so the w–limit set is $\{(1,0)\}$. This means that $\mathcal{P}$ is a domain of attraction of the critical point $(1,0)$. ■

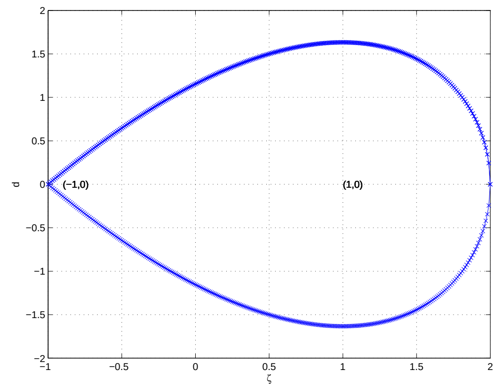
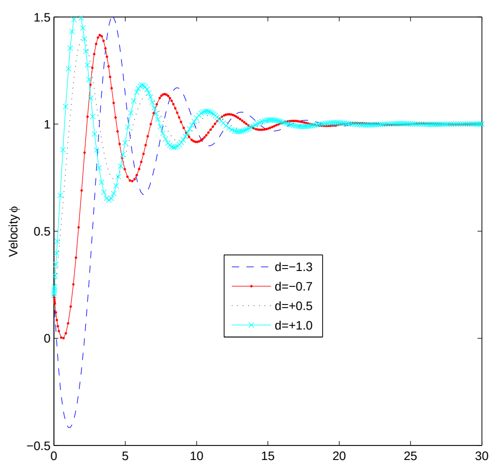


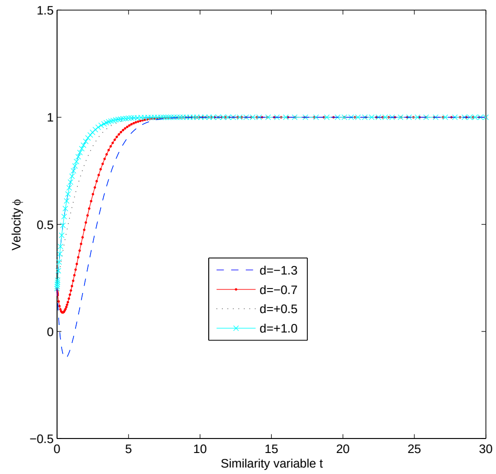
Fig. 3.2 A basin of attraction $\mathcal{P}$ of the critical point $(1,0)$.

4. Numerical results

In this section numerical solutions of the boundary–value problem (3.11) are obtained by using the fourth-order Runge-Kutta integration scheme with the shooting method. velocity profiles of the dimensionless velocity θ are plotted in term of the similarity variable t, for various value of the shooting parameter d (the dimensionless skin-friction), *Fig.*4.1 (1) and (2) show that the numerical results are in good agreement with the above theoretical predictions.



(1)



(2)

Fig. 1.4 Velocity profiles in terms of $d = \theta'(0)$ for fixed $\zeta = 0.2$ and $\tau = 0.5$ (1), and $\tau = 2.83$ (2).

5. Conclusion

In this work the laminar two-dimensional steady incompressible, boundary layer flow past a stretching surface is considered. It has been shown that the problem has solutions having a similarity form if the velocity distribution outside the boundary layer is proportional to x^m, for some real number m. In the second part of this paper, we are interested in question of existence of solutions t in the case where the external velocity is an inverse-linear function; $m = -1$. This situation occurs in the case of sink flow. To obtain exact solutions the stream function ψ is written under the form

$$\psi(x,y) = \sqrt{\mu U_\infty} f(t) + V_w \log(x). \tag{5.1}$$

It is shown that the ordinary differential equation satisfied by f has multiple solutions for any V_w positive (suction) and no solution can exist if $V_w \leq 0$ (injection). A sufficient condition for the existence is derived:

$$\zeta > -1, \qquad \frac{1}{2} f''(0)^2 + \zeta\left(\frac{\zeta^2}{3} - 1\right) < \frac{2}{3}. \tag{5.2}$$

We have obtained two family of solutions according to $\tau = V_w\,(\mu U_\infty)^{-1/2}$. If $\tau \geq \sqrt{8}$, f' is monotonic and goes to unity at infinity, but if $0 < \tau < \sqrt{8}$, we have a stable spiral. The function f' oscillates an infinite number of times and goes to 1. So if we are interested in solutions to (3.9) such that

$$-1 < f' < 1$$

we must take U_w, V_w and $U_\infty > 0$ satisfying $-U_\infty < U_w < U_\infty$ and $V_w > (8\nu U_\infty)^{1/2}$.
Condition (5.2) indicates also that for the same positive value of the suction parameter the permeable wall stretching with velocity $U_w x^{-1}, U_w > 0$ has multiple boundary–layer flows. Every flow is uniquely determined by the dimensionless skin friction $f''(0)$ which can be any real number in the interval

$$\left(-\sqrt{\tfrac{4}{3} + 2\zeta\,(1-\zeta^2/3)}, \sqrt{\tfrac{4}{3} + 2\zeta\,(1-\zeta^2/3)}\right),$$

where $\zeta = U_w U_\infty^{-1}$. The case $U_\infty = 0$ was considered by Magyari, Pop and Keller [27]. The authors showed, by numerical solutions, that the boundary layer flow exists only for a large suction parameter ($\tau \geq 1.079131$).
The existence of exact solutions of the Falkner-Skan equation under the present condition was discussed by Rosenhad [28, pp. 244–246] who mentioned that these results may be obtained by rigorous arguments which, in fact, motivated the present work. We note, in passing, that it is possible to obtain solutions if the the skin friction satisfies $|f''(0)| > \sqrt{\frac{4}{3} + 2\zeta\,(1-\zeta^2/3)}$.

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